

Attainable sets for left invariant control systems and Carnot–Caratheodory metrics on nilpotent Lie groups

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Abstract

Let G be a semidirect product of a nilpotent Lie group and \mathbb{R} . For a left invariant control system on G whose control domain is defined by a convex cone in the Lie algebra of G we prove that the attainable set coincides with a "halfspace" if the degree of contact of the cone with certain linear subspaces of the Lie algebra is sufficiently high.

1 Introduction

Let N be a Lie group and C be a subset of its Lie algebra \mathcal{N} identified with the tangent space to N at the identity e . Denote by $\mathfrak{T}(C)$ the set of all piecewise smooth curves with both one-side tangent vectors in the corresponding left translation of C :

$$\gamma'(t) \in d_e \lambda_{\gamma(t)}(C) \quad \text{where} \quad \lambda_g(h) = gh. \quad (1)$$

The attainable set $\mathfrak{R}(C, p)$ is the closure of endpoints for curves in $\mathfrak{T}(C)$ which start at p . Put $\mathfrak{R}(C) = \mathfrak{R}(C, e)$ where e is the identity. We consider the problem: given C , find $\mathfrak{R}(C)$. In this setting we may assume without loss of generality that C is a closed convex cone. If $\mathfrak{R}(C) = N$ then C is called *controllable*. The opposite property is the globality: put

$$H(C) = \{\xi \in \mathcal{N} : \exp(t\xi) \in \mathfrak{R}(C) \quad \text{for all} \quad t \geq 0\},$$

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then C is called *global* if $H(C) = C$. The group N is supposed to be nilpotent and simply connected. For generating cones in nilpotent Lie algebras the criterion of controllability due to Hilgert, Hofmann, and Lawson [6], is known from early 80-th:

$$\mathfrak{R}(C) = N \quad \text{if and only if} \quad \text{Int } C \cap [\mathcal{N}, \mathcal{N}] \neq \emptyset. \quad (2)$$

If $C \cap [\mathcal{N}, \mathcal{N}] = \{0\}$ then the cone is global. We consider the intermediate case

$$\text{Int}(C) \cap [\mathcal{N}, \mathcal{N}] = \emptyset, \quad C \cap [\mathcal{N}, \mathcal{N}] \neq \{0\}.$$

This situation can be described as follows: there exist a boundary point of C and a supporting hyperplane H at this point which includes $[\mathcal{N}, \mathcal{N}]$. Then the set $\mathfrak{R}(C)$ cannot coincide with N – it is included to a "halfspace"

$$N_\chi^+ = \chi^{-1}(\mathbb{R}^+),$$

where χ is the continuous homomorphism $N \rightarrow \mathbb{R}$ whose tangent homomorphism annihilates H , $\mathbb{R}^+ = [0, \infty)$. We shall prove that $\mathfrak{R}(C) = N_\chi^+$ for some χ if the degree of contact of C with certain subspaces is sufficiently high. The role of the degree of contact was notified in [4] where global Ad-invariant cones were characterized – while the final answer was formulated by the algebraic language, in fact, the globality of an invariant cone in a Lie algebra is determined by the degree of contact of the cone with the linear sum of two distinct nilpotent subalgebras. In this article we use Carnot–Caratheodory metrics to prove the result mentioned above. Probably, these metrics can be a natural and essential tool in Geometric Control Theory, in particular, for the investigation of attainable sets. In any way, the usage of Carnot–Caratheodory metrics clarifies the dependence of $\mathfrak{R}(C)$ on the degree of contact. They also give quantitative versions for the criterion of controllability (2). For a discussion of the role of Lie groups and algebras in Control Theory and further references, see [2], [3].

2 Preliminaries and statement of the result

2.1 Realization of simply connected nilpotent Lie groups. Let \mathcal{N} be a nilpotent Lie algebra. The corresponding Lie group N can be realized as \mathcal{N} with the group multiplication defined by the Campbell–Hausdorff formula:

$$xy = x + y + \frac{1}{2}[x, y] + P_3(x, y) + \dots + P_d(x, y), \quad (3)$$

where $P_k(x, y)$, $k = 3, \dots, d$, is the sum of Lie products of the length k . Thus P_k is a homogeneous polynomial of the degree k with values in \mathcal{N} . It follows from (3) that for all $\xi \in \mathcal{N}$

$$\exp(\xi) = \xi, \quad \xi^{-1} = -\xi, \quad (4)$$

$e = 0$, and the multiplicative commutator has the form

$$\{x, y\} = xyx^{-1}y^{-1} = [x, y] + \text{(Lie products of the length > 2)} \quad (5)$$

We shall consider simultaneously the Lie group and the Lie algebra structures. In particular, we keep the vector notation for addition and multiplication by scalars. It will be convenient to fix the euclidean distance in \mathcal{N} which will be denoted by $|\cdot|$. Thus, for example,

$$x^n = nx \quad \text{and} \quad |x^n| = |n||x| \quad \text{for all integer } n. \quad (6)$$

Put

$$\mathcal{N}^1 = \mathcal{N}, \quad \mathcal{N}^{k+1} = [\mathcal{N}, \mathcal{N}^k]; \quad \mathcal{N}^1 \supset \mathcal{N}^2 \supset \dots \supset \mathcal{N}^d \supset \mathcal{N}^{d+1} = \{0\}, \quad (7)$$

where $\mathcal{N}^d \neq \{0\}$. Then \mathcal{N}^k , $k = 1, \dots, d$ is also a normal subgroup of \mathcal{N} which sometimes will be denoted by N^k . For each k chose in \mathcal{N}^k a complementary to \mathcal{N}^{k+1} subspace \mathcal{N}_k . Then \mathcal{N} is the linear direct sum of these subspaces

$$\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2 \oplus \dots \oplus \mathcal{N}_d \quad (8)$$

and \mathcal{N}_1 generates \mathcal{N} as a Lie algebra.

2.2 Graded nilpotent Lie algebras. If $[\mathcal{N}_k, \mathcal{N}_l] \subseteq \mathcal{N}_{k+l}$ then (8) is the gradation of \mathcal{N} which satisfies the additional condition

$$[\mathcal{N}_1, \mathcal{N}_l] = \mathcal{N}_{l+1}, \quad l = 1, \dots, d. \quad (9)$$

which is equivalent to the assumption that \mathcal{N}_1 generates \mathcal{N} . Let us pick $x \in \mathcal{N}$, decompose it according to (8)

$$x = x_1 + \dots + x_d,$$

and put

$$Dx = x_1 + 2x_2 + \dots + dx_d.$$

If (8) is the gradation then D is a differentiation of \mathcal{N} and

$$\delta_t : x_1 + x_2 + \dots + x_d \rightarrow tx_1 + t^2x_2 + \dots + t^dx_d, \quad t > 0 \quad (10)$$

is the corresponding one-parametrical group of automorphisms written in the multiplicative form. Since \exp is identical, δ_t is also the isomorphism of the group N . Further, the group $\{\delta_t\}_{t \geq 0}$ can be extended to the complexification of \mathcal{N} and nonzero complex values of t . This implies that

$$\delta_{-1} : x_1 + x_2 + \dots + x_d \rightarrow -x_1 + x_2 + \dots + (-1)^k x_k + \dots + (-1)^d x_d \quad (11)$$

is an isomorphism of \mathcal{N} and N .

2.3 Asymptotic group. If (8) is not a gradation then the formula

$$[x, y]^a = \lim_{t \rightarrow \infty} \delta_t^{-1}[\delta_t x, \delta_t y] \quad (12)$$

defines the asymptotic Lie product in \mathcal{N} and (8) is the gradation for it. This gradation also can be defined by the standard factorization procedure for the filtration (7): the new Lie bracket for $x \in \mathcal{N}_k$, $y \in \mathcal{N}_l$ is the projection of the old one in \mathcal{N}^{k+l} to \mathcal{N}_{k+l} along \mathcal{N}^{k+l+1} . Indeed, let

$$[x, y] = [x, y]_1 + \dots + [x, y]_d, \quad x = x_1 + \dots + x_d, \quad y = y_1 + \dots + y_d$$

be corresponding to (8) decompositions; then

$$\delta_t^{-1}[\delta_t x, \delta_t y] = \sum_{p \geq k+l} t^{k+l-p} [x_k, y_l]_p = \sum_{p=k+l} [x_k, y_l]_p + \alpha(t), \quad (13)$$

where $|\alpha(t)| = O(\frac{1}{t})$, and the limit can be easily calculated in this notation. The corresponding group N_a also can be realized as a limit. For each $t > 0$, put

$$x \cdot_t y = \delta_t^{-1}(\delta_t x \cdot \delta_t y).$$

This introduces in \mathcal{N} the structure of a Lie group isomorphic to N . It follows from the Campbell-Hausdorff formula and (13) that for every $x, y \in N$ there exists the limit

$$x \cdot_a y = \lim_{t \rightarrow \infty} x \cdot_t y \quad (14)$$

Moreover, by (13) and (3)

$$x \cdot_t y = \delta_t^{-1}(\delta_t x \cdot \delta_t y) = x \cdot_a y + \beta(x, y, t), \quad (15)$$

where

$$|\beta(x, y, t)| \leq \frac{A}{t} (|x| + |x|^d)(|y| + |y|^d)$$

and $A > 0$ depends only on the algebra \mathcal{N} .

2.4 A construction for curves in $\mathfrak{T}(C)$. Let $\gamma_1 : [0, a_1] \rightarrow N$ and $\gamma_2 : [0, a_2] \rightarrow N$ be paths in N starting at e : $\gamma_k(0) = e$, $k = 1, 2$. Put

$$\gamma_1 \cdot \gamma_2(t) = \begin{cases} \gamma_1(t), & t \in [0, a_1] \\ \gamma_1(a_1)\gamma_2(t - a_1), & t \in [a_1, a_1 + a_2] \end{cases}$$

Then, for any $C \subseteq \mathcal{N}$, $\gamma_1 \cdot \gamma_2$ belongs to $\mathfrak{T}(C)$ if so are γ_1 and γ_2 . An important particular case of this construction is as follows: for $\xi \in \mathcal{N}$ put $\bar{\xi}(t) = \exp(t\xi)$, $t \in [0, 1]$, and

$$\bar{x} = \bar{\xi}_1 \cdot \bar{\xi}_2 \cdot \dots \cdot \bar{\xi}_n, \quad \text{where } x = (\xi_1, \dots, \xi_n), \quad \xi_1, \dots, \xi_n \in C. \quad (16)$$

Then $\bar{x}(n) = \exp(t\xi_1) \dots \exp(t\xi_n)$. Furthermore, if $\xi_1, \dots, \xi_n \in C$ then $\bar{x} \in \mathfrak{T}(C)$. For the Riemannian left invariant metric defined by the euclidean norm $\| \cdot \|$

$$\Lambda(\bar{x}) = \Lambda(\bar{\xi}_1) + \dots + \Lambda(\bar{\xi}_n), \quad (17)$$

where $\Lambda(\gamma)$ denotes the length of the curve γ . The length of the curve $\bar{\xi}$, $\xi \in C$, can be easily derived: $\Lambda(\bar{\xi}) = |\xi|$. Hence for $\xi_1, \dots, \xi_n \in C$

$$\Lambda(\bar{x}) = |\xi_1| + \dots + |\xi_n|. \quad (18)$$

Clearly, each curve in $\mathfrak{T}(C)$ can be approximated by curves of the type (16).

2.5 Carnot–Caratheodory metrics. Recall the definition of Carnot–Caratheodory metrics which also are known as subriemannian or nonholonomic Riemannian ones. Any euclidean norm $\| \cdot \|$ on \mathcal{N}_1 uniquely determines the left invariant norm on the left invariant distribution of subspaces generated by \mathcal{N}_1 . Hence the length of a curve $\gamma \in \mathfrak{T}(\mathcal{N}_1)$ can be defined by the standard formula

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)|_{\gamma(t)} dt. \quad (19)$$

Since $\bar{\xi} \in \mathfrak{T}(\mathcal{N}_1)$ for $\xi \in \mathcal{N}_1$, the left invariance of the Carnot–Caratheodory metric implies that (17) and (18) are true for curves described in Subsection 2.4 with $C = \mathcal{N}_1$. The Carnot–Caratheodory distance $\kappa(x, y)$ between x and y is defined as the least lower bound for lengths of curves in $\mathfrak{T}(\mathcal{N}_1)$ which join x and y :

$$\kappa(x, y) = \inf\{\Lambda(\gamma) : \gamma \in \mathfrak{T}(\mathcal{N}_1), \gamma : [a, b] \rightarrow N, \gamma(a) = x, \gamma(b) = y\}. \quad (20)$$

Each piecewise smooth curve in \mathcal{N}_1 has the unique lift to the curve in $\mathfrak{T}(\mathcal{N}_1)$. Hence the natural projection $\pi_k : N \rightarrow N/N^k$, $k = 2, \dots, d$, keeps the length of a curve γ in $\mathfrak{T}(\mathcal{N}^1)$ which is equal to the usual euclidean length of $\pi_2\gamma$:

$$\Lambda(\pi_2\gamma) = \Lambda(\pi_3\gamma) = \dots = \Lambda(\pi_d\gamma) = \Lambda(\gamma). \quad (21)$$

If (8) is the gradation satisfying (9) then δ_t is an automorphism, hence it commutes with the lifting procedure. Therefore, δ_t in (10) is a *metric dilation* for $t > 0$, i.e.

$$\rho(\delta_t(x), \delta_t(y)) = t\rho(x, y), \quad x, y \in N. \quad (22)$$

Since $\delta_t(\xi) = t\xi$ for $\xi \in \mathcal{N}_1$, (22) follows from (19) and the definition of κ . Further, (19) and (11) implies that δ_{-1} is an isometry:

$$\kappa(\delta_{-1}(x), \delta_{-1}(y)) = \kappa(x, y) \quad \text{for all } x, y \in N. \quad (23)$$

If \mathcal{N} is not graded then δ_t is not an automorphism but this is true for the limit group defined by (14). It can be equipped with the limit metric

$$\kappa_a(x, y) = \lim_{t \rightarrow \infty} \kappa_t(x, y), \quad \text{where } \kappa_t(x, y) = \frac{1}{t} \kappa(\delta_t x, \delta_t y). \quad (24)$$

The asymptotic group could be realized as the Gromov–Hausdorff limit of metric spaces – groups with left invariant metrics κ_t . For more details on this subject, see ([5]), ([9]).

2.6 Inner metrics. We use a definition of the inner metric which is equivalent to the standard one in the class of left invariant metrics on Lie groups (see [1]). Let ρ be a left invariant metric which is compatible with the topology, $B(x, r)$ denote the open ball at x of radius $r > 0$, $B(x, 0) = \{x\}$. Put $B(r) = B(e, r)$ and $B(0) = \{e\}$. The left invariance of the metric ρ means that

$$B(x, r) = xB(r).$$

We shall say that ρ is *inner* if

$$B(r)B(s) = B(r + s) \quad \text{for all } r, s \geq 0. \quad (25)$$

The product of sets is taken pointwise: $AB = \{ab : a \in A, b \in B\}$. The same equality is true for closed ball since they are compact. Taken together with the left invariance, this implies for any two points in N the existence of the shortest curve which joins them. If H is a normal closed subgroup of a Lie group G with the inner metric ρ and $\pi : G \rightarrow G/H$ is the canonical projection then $\tilde{B}(r) = \pi B(r)$ is the unit ball for the metric

$$\tilde{\rho}(\tilde{x}, \tilde{y}) = \inf\{\rho(xh_1, yh_2) : h_1, h_2 \in H\}. \quad (26)$$

Clearly, this metric is inner because (25) is satisfied for it. Furthermore, Riemannian and Carnot–Caratheodory metrics are inner – this follows from (20) for Carnot–Caratheodory metrics and from the analogous formula for

Riemannian ones. The identity (25), in particular, implies that all inner metrics are equivalent "in large": for each pair of inner metrics ρ, ρ' which define the same topology and any $\varepsilon > 0$, there exist $C, c > 0$ such that

$$x, y \in G, \rho(x, y) > \varepsilon \implies c < \frac{\rho'(x, y)}{\rho(x, y)} < C \quad (27)$$

Indeed, for some $C, c > 0$ inclusions $B(c\varepsilon) \subseteq B'(\varepsilon) \subseteq B(C\varepsilon)$ holds, where $B'(\varepsilon)$ is the ρ' -ball. By (25),

$$B(nc\varepsilon) \subseteq B'(n\varepsilon) \subseteq B(nC\varepsilon)$$

for all positive integer n , and the left invariance of these metrics implies (27). Thus, each inner metric asymptotically (for great distances) equivalent to a Carnot–Caratheodory metric. Note that the Carnot–Caratheodory metric for the asymptotic group is self-similar – it admits metric dilations. Left invariant inner metrics on topological groups were studied in ([7]), ([8]). For Lie groups, they are Finsler (maybe nonholonomic) ones.

2.7 Degree of contact. Let η be an increasing function defined on some interval in \mathbb{R} with the left endpoint 0, $\lim_{\varepsilon \rightarrow 0} \eta(\varepsilon) = 0$, \mathcal{L} be a linear subspace of the Euclidean space \mathcal{N} . We shall say that a cone C has the *degree of contact with \mathcal{L} at the point $x \in C$ greater than η* if

$$\text{dist}(C, x + y) = o(\eta(|y|)) \quad \text{as } y \rightarrow 0 \quad \text{in } \mathcal{L} \quad (28)$$

The degree of contact of C with \mathcal{L} at x is *greater or equal to η* if there exist $Q > 0$ and a neighborhood U of zero in \mathcal{L} such that

$$Q\eta(|y|) \geq \text{dist}(C, x + y) \quad \text{for all } y \in U$$

Suppose that $x \notin \mathcal{L}$; then the degree of contact is *equal* to η if it is greater or equal and the inverse inequality holds with some another constant. If $x \in \mathcal{L}$ then one has to replace \mathcal{L} in this definition to any subspace $\mathcal{L}' \subset \mathcal{L}$ complementary to $\mathbb{R}x$ in \mathcal{L} , the definition doesn't depend on the choice of \mathcal{L}' .

If $\eta(\varepsilon) = \varepsilon^a$ then a will be called the degree of contact and denoted by $\text{cont}(C, \mathcal{L}, x)$; $\text{cont}(C, \mathcal{L}, x) > a$ ($\text{cont}(C, \mathcal{L}, x) \geq a$, $\text{cont}(C, \mathcal{L}, x) = a$) will mean that the degree of contact is greater than (respectively, greater or equal to, equal to) ε^a . For example, the degree of contact of any Lorentzian cone with each its tangent hyperplane is equal to 2: $\text{cont}(C, T_x \partial C, x) = 2$ for any $x \in \partial C$.

2.8 Statement of the main result. The result is proved for slightly more general setting than it was mentioned above: the group need not be nilpotent in general – it is supposed to be a semidirect product of \mathbb{R} and a nilpotent group. We keep the notation for \mathcal{N} , N , particularly (7), which were introduced above. Let \mathbb{R}^+ denote the set of all nonnegative real numbers. By \mathcal{G} we denote a real Lie algebra of a simply connected Lie group G which is a semidirect product of \mathbb{R} and a nilpotent group N , $\chi : G \rightarrow \mathbb{R}$ be the projection homomorphisms to the factor \mathbb{R} . Set $G^+ = \chi^{-1}(\mathbb{R}^+)$, and $\mathcal{G}^+ = d_e^{-1}\chi(\mathbb{R}^+)$ (hence $\exp(\mathcal{G}^+) = G^+$).

Theorem 1 *Let $C \subseteq \mathcal{G}^+$ be a convex closed generating cone, $d > 1$, $p \in \partial C \cap \mathcal{N}^d$; further, suppose that there exists $v \in \mathcal{Z}(p) \cap \text{Int}(C) \neq \emptyset$ admitting $\text{ad}(v)$ -invariant linear subspace $\mathcal{N}_1 \subset \mathcal{N}^1$ complementary to \mathcal{N}^2 , and*

$$\text{cont}(C, \mathcal{N}_1, p) > \frac{d}{d-1}. \quad (29)$$

Then $\mathfrak{R}(C) = G^+$.

3 Quantitative versions of the controllability

In the following two lemmas the Lie algebra \mathcal{N} is supposed to be graded as in (8), (9).

Lemma 1 *Let ρ be an inner metric in N , $z \in N_d$, $\varepsilon > 0$, and $r = \rho(e, z)$. Then*

$$e \in B(z, \varepsilon)^n \quad \text{if and only if} \quad n > \left(\frac{r}{\varepsilon}\right)^{\frac{d}{d-1}}. \quad (30)$$

Moreover, for any $s > 0$, if

$$n^{\frac{1}{d}}r + s < n\varepsilon \quad (31)$$

then $B(z, \varepsilon)^n \supset B(s)$.

Proof. Since z belongs to the center of N ,

$$B(z, \varepsilon) = zB(\varepsilon) = B(\varepsilon)z. \quad (32)$$

Taken together with (25) and (6), (32) implies that

$$B(z, \varepsilon)^n = (zB(\varepsilon))^n = (nz)B(n\varepsilon) = B(nz, n\varepsilon). \quad (33)$$

Therefore, the inclusion in the left side of (30) is equivalent to the inequality

$$\rho(e, nz) < n\varepsilon. \quad (34)$$

By (22) and (10),

$$\rho(e, nz) = n^{\frac{1}{d}}\rho(e, z) = n^{\frac{1}{d}}r.$$

Thus inequality (33) holds if and only if

$$n^{\frac{1}{d}}r < n\varepsilon.$$

This is equivalent to the right part of (30).

For $w \in N$, the assumption $\rho(w, nz) < n\varepsilon$ is equivalent to $w \in B(nz, n\varepsilon)$. By the triangle inequality,

$$\rho(e, nz) < t, \quad \rho(e, w) < s \implies w \in B(nz, t + s),$$

hence the inclusion $B(nz, t + s) \supseteq B(s)$. Put $t = n^{\frac{1}{d}}r$. Then, according to (33), (34), and (31), we receive the desired inclusion. •

Corollary 1 *For non-graded N , if $s > 0$ then there exists $C > 0$ such that for all $\varepsilon > 0$*

$$C(n^{\frac{1}{d}}r + s) < n\varepsilon \quad (35)$$

implies $B(z, \varepsilon)^n \supset B(s)$.

Proof. This follows from the existence of the asymptotic metric (24) for which the assertion holds by the lemma, and (27). •

Lemma 2 *Let $x \in N_{d-1}$, $\varepsilon > 0$. Suppose that $B(x, \varepsilon)^n \cap N_d \neq \{0\}$. Then $e \in B(x, 2\varepsilon)^{2n}$.*

Proof. Since the metric is left invariant, $B(x, \varepsilon) = xB(\varepsilon)$. Let

$$z = xy_1xy_2 \dots xy_n \in N^d$$

where $y_1, y_2 \dots y_n \in B(\varepsilon)$. If d is odd then $\delta_{-1}x = x$, and $\delta_{-1}z = -z = z^{-1}$. Hence

$$e = zz^{-1} = xy_1 \dots xy_n x \delta_{-1}(y_1) xy_2 \dots x \delta_{-1}(y_n) \in B(x, \varepsilon)^{2n}.$$

If d is even then $\delta_{-1}x = x^{-1} = -x$, and $\delta_{-1}z = z$. Therefore,

$$e = z \delta_{-1}(z^{-1}) = xy_1 \dots xy_n \delta_{-1}(y_n) x \dots \delta_{-1}(y_1) xe \in B(x, 2\varepsilon)^{2n}$$

since $y_n \delta_{-1}(y_n) \in B(2\varepsilon)$. •

Corollary 2 *Let $x \in \mathcal{N}^{d-1}$, $\varepsilon > 0$. Then*

$$n > 2 \left(\frac{2r}{\varepsilon} \right)^{\frac{d}{d-1}} \implies e \in B(x, \varepsilon)^n. \quad (36)$$

Proof. Applying (30) to the factor group N/N^d we receive

$$n > \left(\frac{r}{\varepsilon} \right)^{\frac{d}{d-1}} \implies B(x, \varepsilon)^n \cap N^d \neq \emptyset,$$

and Lemma 1, with ε replaced by $\frac{\varepsilon}{2}$, implies the desired inclusion. •

There is a natural way to realize any finite dimensional nilpotent Lie algebra \mathcal{N} as a factor algebra of a finite dimensional graded Lie algebra $\widetilde{\mathcal{N}}$. Let x_1, \dots, x_l be a set of generators for \mathcal{N} (the linear basis of \mathcal{N}_1). Then \mathcal{N} is the homomorphic image of the free Lie algebra \mathcal{F} generated by x_1, \dots, x_l . The kernel of the homomorphism includes the ideal \mathcal{F}^d generated by all products of length $> d$. This means that \mathcal{N} is the homomorphic image of the finite dimensional Lie algebra $\widetilde{\mathcal{N}} = \mathcal{F}/\mathcal{F}^d$ whose natural gradation satisfies the condition (9). Let $\pi : \widetilde{\mathcal{N}} \rightarrow \mathcal{N}$ denote this homomorphism. Clearly, $\pi \widetilde{\mathcal{N}}^k = \mathcal{N}^k$ for $k = 1, \dots, d$. Note that $\widetilde{\mathcal{N}}$ has the same height d and that generating spaces \mathcal{N}_1 and $\widetilde{\mathcal{N}}_1$ may be identified. Thus the euclidean norm $\|\cdot\|$ in \mathcal{N}_1 defines Carnot–Caratheodory metrics κ and $\tilde{\kappa}$ in $N = \mathcal{N}$ and $\widetilde{N} = \widetilde{\mathcal{N}}$ respectively. We shall equip with \sim symbols denoting objects in $\widetilde{\mathcal{N}}$ corresponding to objects in \mathcal{N} . Put

$$\kappa(x) = \kappa(e, x), \quad \tilde{\kappa}(\tilde{x}) = \tilde{\kappa}(\tilde{e}, \tilde{x}).$$

Clearly, $\kappa(\pi \tilde{x}) \leq \tilde{\kappa}(\tilde{x})$, $\tilde{x} \in \widetilde{N}$. In the following theorem we do not assume that \mathcal{N} is graded but keep the notation of the previous section.

Theorem 2 *Let $d > 2$, $k = d$ or $k = d - 1$, $x \in N^k \setminus N^{k+1}$, and $r = \kappa(x)$. Then there exists $Q \geq 1$ such that for all $\varepsilon > 0$ the condition*

$$n > Q \left(\frac{r}{\varepsilon} \right)^{\frac{k}{k-1}} \quad (37)$$

implies that $e \in B(x, \varepsilon)^n$, where $B(x, \varepsilon)$ is the ball for the Carnot–Caratheodory metric κ .

Proof. If \mathcal{N} is graded as in (8), (9), then the assertion of the theorem is an easy consequence of Lemma 1 for $k = d$ and Corollary 2 for $k = d - 1$. In

general case, let us realize \mathcal{N} as the factor algebra of the graded Lie algebra $\tilde{\mathcal{N}}$ by the construction described above. Let $\pi\tilde{x} \in \tilde{\mathcal{N}}^k \setminus \tilde{\mathcal{N}}^{k+1}$, $\pi\tilde{x} = x$, and put $\tilde{r} = \tilde{\kappa}(\tilde{x})$, $K = \frac{\tilde{r}}{r}$. Then, by Lemma 1 or Corollary 2, there exists $A > 0$ such that

$$n > A \left(\frac{\tilde{r}}{\varepsilon} \right)^{\frac{k}{k-1}} \quad \text{implies} \quad \tilde{e} \in \tilde{B}(\tilde{x}, \varepsilon)^n.$$

Since $\pi\tilde{B}(\tilde{x}, \varepsilon)^n = B(x, \varepsilon)^n$, the inclusion $e \in B(x, \varepsilon)^n$ is true for

$$Q = \max\{1, K^{\frac{k}{k-1}} A\} \dots \bullet$$

The following theorem is in fact a reformulation of Theorem 2 by another words. Let ρ be the Riemannian metric defined by the euclidean norm $|\cdot|$ in \mathcal{N} and κ be the Carnot–Caratheodory metric corresponding to the restriction of this norm to \mathcal{N}_1 . Put $\mathcal{B}(r) = \{\xi \in \mathcal{N} : |\xi| < r\}$

Theorem 3 *Let d, k, x, ρ be as above, and let $r = \rho(e, x)$. Then there exists $P > 0$ such that for all sufficiently small $\varepsilon > 0$ the group N admits a closed curve $\gamma \in \mathfrak{T}(x + \mathcal{B}(\varepsilon))$ whose ρ -length satisfies the inequality*

$$\Lambda_\rho(\gamma) \leq P \left(\frac{r}{\varepsilon} \right)^{\frac{k}{k-1}}. \quad (38)$$

Proof. Put $R = |x|$; clearly, $r \leq R \leq \kappa(x)$. If $\varepsilon > R$ then the assertion is evident. Hence we may assume that $\varepsilon < R$. Then there exists $a \in (0, 1)$ such that $x + \mathcal{B}(\varepsilon)$ includes the ρ -ball at x of radius $a\varepsilon$ for all $\varepsilon \in (0, R)$ (recall that we identify \mathcal{N} and N). Hence it includes the κ -ball $B(x, a\varepsilon)$. Let Q be as in Theorem 2. Then there exists integer n which satisfies inequalities

$$Q \left(\frac{\kappa(x)}{a\varepsilon} \right)^{\frac{k}{k-1}} < n \leq 2Q \left(\frac{\kappa(x)}{a\varepsilon} \right)^{\frac{k}{k-1}}.$$

Then, by the first of them and Theorem 2, there exist $x_1, \dots, x_n \in B(x, a\varepsilon)$ such that $x_1 \dots x_n = e$. Since \exp is identical, it follows from the construction of Subsection 2.4 that $\bar{x}_1, \dots, \bar{x}_n \in \mathfrak{T}(B(x, a\varepsilon))$ and the curve $\gamma = \bar{x}_1 \dots \bar{x}_n$ is closed. By (18),

$$\Lambda_\rho(\gamma) = |x_1| + \dots + |x_n| \leq 2Rn < P \left(\frac{r}{\varepsilon} \right)^{\frac{k}{k-1}},$$

where

$$P = 4QR \left(\frac{\kappa(x)}{ar} \right)^{\frac{k}{k-1}}.$$

This proves the theorem. \bullet

4 Attainable sets

Everywhere in this section we suppose that the assumption of Theorem 1 are satisfied. Let \mathcal{G} be equipped with the euclidean norm $\| \cdot \|$ and G with the corresponding left invariant Riemannian metric ρ and N with Carnot–Caratheodory metric κ . Then the semidirect product $G = \mathbb{R} \ltimes N$ is defined by the one-parametrical group $A_t = e^{t \text{ad}(v)}$, $t \in \mathbb{R}$, of group automorphisms of N . Since \exp is identical for the coordinate system in N which we use, A_t is also the one-parametrical group of automorphisms of \mathcal{N} . Hence A_t is linear in these coordinates. Put

$$M = \sup\{\|A_t\| : |t| \leq 1\}. \quad (39)$$

The multiplication law in the group G can be written explicitly:

$$(t, x)(s, y) = (t + s, (A_{-s}x)y), \quad \text{where } x, y \in N, t, s \in \mathbb{R}.$$

We denote $v = (1, 0)$. By the assumption of the theorem,

$$p + v \in \text{Int}(C), \quad (40)$$

$$v \in \mathcal{Z}(p). \quad (41)$$

Set

$$\mathcal{B} = \{\xi \in \mathcal{G} : |\xi| < 1\}, \quad \mathcal{B}_1 = \mathcal{B} \cap \mathcal{N}_1,$$

and let $B_\kappa(r)$ the κ -ball with the center e of the radius r . In the following lemma we consider these sets as subsets of the group N .

Lemma 3 *For any $r > 0$*

$$B_\kappa(r) = \bigcup_{n=1}^{\infty} \left(\frac{r}{n} \mathcal{B}_1\right)^n. \quad (42)$$

Proof. For each $x \in \mathcal{B}_1$ the curve \bar{x} belongs to $\mathfrak{T}(\mathcal{B}_1)$. Hence $\frac{r}{n} \mathcal{B}_1 \subseteq B_\kappa(\frac{r}{n})$. By (25), the left side of the equality includes the right one. Let $\gamma : [0, r] \rightarrow N$ be a curve in $\mathfrak{T}(\mathcal{B}_1)$. It follows from the definition of Carnot–Caratheodory metric that the open ball $B_\kappa(r)$ is filled by points $\gamma(r)$ for such curves γ . Let $\lambda_g(h) = gh$ be the left shift by g . The endpoint of the curve

$$\gamma_n = \bar{x}_1 \cdot \dots \cdot \bar{x}_n,$$

where

$$x_k = \frac{1}{n} d_{\gamma(t_k)} \lambda_{\gamma(t_k)}^{-1}(\gamma'(t_k)), \quad t_k = \frac{rk}{n}, \quad k = 1, \dots, n,$$

belongs to the right side of (42). Clearly, $\gamma_n(t) \rightarrow \gamma(t)$ as $n \rightarrow \infty$ for each $t \in [0, r]$. Hence the right side of (42) is dense in the left one. Further, it follows from (5) and (4) that $(s\mathcal{B}_1)^k$ is open for sufficiently large k depending only on \mathcal{N} . Therefore, the right side of (42) is open; let us denote it by $\tilde{B}(r)$. By standard arguments it is not difficult to show that

$$\bigcup_{n=1}^{\infty} \left(\frac{r}{n}\mathcal{B}_1\right)^n = \bigcup_{n=1}^{\infty} \left(\frac{r}{2^n}\mathcal{B}_1\right)^{2^n}.$$

Hence $\tilde{B}(\frac{r}{2})^2 = \tilde{B}(r)$, and this division procedure can be continued. This implies that $\tilde{B}(r)$ coincides with the interior of its closure. Since $\tilde{B}(r)$ is dense in $B_{\kappa}(r)$ and open, $\tilde{B}(r) = B_{\kappa}(r)$. •

Lemma 4 *If $q \in N^d$, $\varepsilon > 0$, $t > 0$, $n \in \mathbb{N}$, and $nt < 1$ then*

$$(t, q + M\varepsilon\mathcal{B}_1)^n \supseteq (nt, nq + (\varepsilon\mathcal{B}_1)^n); \quad (43)$$

$$(t, B_{\kappa}(q, M\varepsilon))^n \supseteq (nt, B_{\kappa}(q, \varepsilon)^n). \quad (44)$$

Proof. Let $x_1, \dots, x_n \in (p + \varepsilon\mathcal{B}_1)$. Put $\tilde{x}_k = (t, A_{(n-k)t}x_k)$. Then

$$\begin{aligned} \tilde{x}_1\tilde{x}_2 \dots \tilde{x}_n &= (t, A_{(n-1)t}x_1)(t, A_{(n-2)t}x_2) \dots (t, x_n) = \\ &= (2t, A_{(n-2)t}(x_1x_2)) \dots (t, x_n) = \dots = (nt, x_1x_2 \dots x_n). \end{aligned}$$

Since \mathcal{N}_1 is A_t -invariant, $nt < 1$, by (41) and (39), $A_{(n-k)t}x_k \in (p + M\varepsilon\mathcal{B}_1)$ for all $k = 1, \dots, n$. To prove (43), it remains to note that $(q + \varepsilon\mathcal{B}_1)^n = nq + (\varepsilon\mathcal{B}_1)^n$ because p belongs to the center of N . The inclusion (44) follows from the same equality, with $x_k \in B_{\kappa}(q, \varepsilon)$, and the inequality $\kappa(e, A_t x) \leq M\kappa(e, A_t x)$ which is an easy consequence of the definition Carnot–Caratheodory metric κ . •

The following elementary lemma whose assertion could be a definition of the degree of contact was already proved in ([4]). We omit the proof – it is rather long than hard. Let \mathcal{L} be as in Subsection 2.7. Put

$$\mathcal{B}_{\mathcal{L}} = \{y : y \in \mathcal{L}, |y| \leq 1\}.$$

Lemma 5 *Let C be a generating closed cone in \mathcal{N} , $x \in C$, $x \neq 0$. Suppose that $\text{cont}(C, \mathcal{L}, x) > a \geq 1$, $v \in \mathcal{E}$, and $x + v \in \text{Int}(C)$. Then there exists a function φ defined on some interval $(0, \alpha)$, $\alpha > 0$, such that $\varphi(\varepsilon) = o(\varepsilon^a)$ as $\varepsilon \rightarrow 0$ and*

$$x + \varphi(\varepsilon)v + \varepsilon\mathcal{B}_{\mathcal{L}} \subset \text{Int}(C)$$

for all $\varepsilon \in (0, \alpha)$. •

Proof of Theorem 1. Let $B_\kappa(p, \varepsilon)$ be the Carnot–Caratheodory ball in N , $B_\kappa = B(e, 1)$, α , φ be as in Lemma 5 with $a = \frac{d}{d-1}$, $\mathcal{L} = \mathcal{N}_1$. There exists a function ψ such that

$$\lim_{t \rightarrow 0} \frac{\varphi(t)}{\psi(t)} = 0, \quad (45)$$

$$\psi(t) = o(t^{\frac{d}{d-1}}). \quad (46)$$

It follows from Corollary 1 that there exists $A > 0$ such that

$$n > A\varepsilon^{-\frac{d}{d-1}} \Rightarrow B_\kappa(p, \varepsilon)^n \supseteq B_\kappa.$$

For these n and sufficiently small $\varepsilon > 0$, applying (44) we receive

$$(\psi(\varepsilon), B_\kappa(p, M\varepsilon))^n \supseteq (n\psi(\varepsilon), B_\kappa(p, \varepsilon)^n) \supseteq (n\psi(\varepsilon), B_\kappa).$$

By (46), since n can be chosen satisfying the inequality $n < 2A\varepsilon^{-\frac{d}{d-1}}$, the set $\mathfrak{R}(C)$ includes the ball B_κ , hence the group N and the halfspace G^+ . Therefore, it is sufficient to prove the inclusion

$$\mathfrak{R}(C) \supset (\psi(\varepsilon), B_\kappa(p, M\varepsilon)) \quad (47)$$

for $\varepsilon \in (0, \alpha)$ for some $\alpha > 0$. It follows from Lemma 5 and (45) that

$$C \supset (\psi(\varepsilon)v, p + M^2\varepsilon\mathcal{B}_1)$$

if ε is sufficiently small. Since C is a cone,

$$C \supset \frac{1}{n}(\psi(\varepsilon)v, p + M^2\varepsilon\mathcal{B}_1), \quad n \in \mathbb{N},$$

hence

$$\mathfrak{R}(C) \supset \left(\frac{1}{n}(\psi(\varepsilon), p + M^2\varepsilon\mathcal{B}_1) \right)^n \supseteq \left((\psi(\varepsilon), p + (\frac{M\varepsilon}{n}\mathcal{B}_1)^n) \right)$$

by (43), and the desired inclusion (47) follows from Lemma 3. •

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